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COMPUTATIONS OF CHOW RINGS AND THE MOD p MOTIVIC COHOMOLOGY OF CLASSIFYING SPACES

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ABSTRACT. In this note, we explain how to compute mod p motivic cohomology over C, the complex number field, by only using algebraic topology. Examples of algebraic spaces X are classifying spaces BG of algebraic groups.

1. CHOW RING, MILNOR K-THEORY, ÉTALE COHOMOLOGY

We use some category Spc of (algebraic) spaces , defined by Voevodsky, where schems A, quotients A_1/A_2 and $colim(A_\alpha)$ are all contained ([Vo2],[Mo-Vo]). Here schemes are defined over a field k with $ch(k)=0$. The motivic cohomology is the double indexed cohomology defined by Suslin and Voevodsky directely related with the Chow ring, Milnor K-theory and étale cohomology,

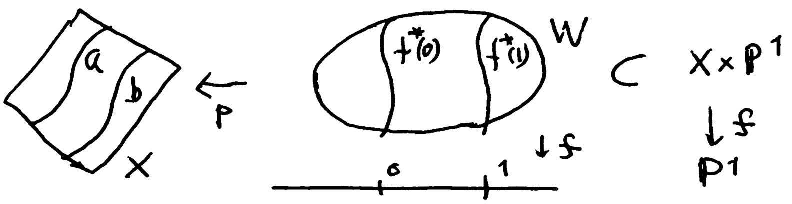
(CH) For a smooth scheme X, $H^{2n,n}(X) = CH^n(X)$: the classical Chow group.

(MK) $H^{n,n}(Spec(k)) \cong K_n^M(k)$, the Milnor K-group for the field k.

For a smooth variety X of $dim(X)=n$. The Chow ring is the sum $CH^*(X) = \oplus_i CH^i(X)$ where

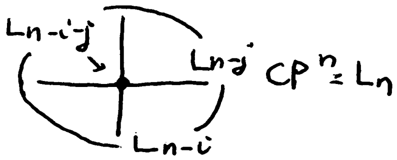
$$CH^i(X) = \{(n-i)cycles\ in\ X\}/(rational\ equivalence).$$

Here the rational equivalence $a \equiv b$ is defiend if there is a codimension i subvariety W in $X \times \mathrm{P}^1$ such that $a = p_* f^*(0)$ and $b = p_* f^*(1)$ where P^1 is the projective line, p(resp. f) is the projection for the first (resp. second) factor.



The multiplications in $CH^*(X)$ is giving by intersections of cycles. Let $k = \mathrm{C}$. Let P^n be the n-dimensional projective space. Then $CH^i(\mathrm{P}^n) \cong \mathrm{Z}\{L_{n-i}\}$ where $L_{n-i} \cong \mathrm{P}^{n-i}$ is an $n-i$-dimensionalsubspace of P^n. Hence the product is $L_{n-i}.L_{n-j} = L_{n-i-j}$. This shows that

$$CH^*(\mathrm{P}^n) \cong \mathrm{Z}[y]/(y^{n+1}) \cong H^*(C\mathrm{P}^n)$$



identifying $y^i = L_{n-i}$.

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Since Spc contains $colimit$, we can consider the infinite projective space $\mathrm{P}^{\infty} = BG_m$ and the infinite Lens spce $colim_n(\mathrm{A}^n - \{0\}/Z/p) = L_p^{\infty} = BZ/p$. The Chow rings of BZ/p are are given in [To 1]

$$(1.1)\quad CH^*(\mathrm{P}^{\infty}) \cong H^{2*,*}(\mathrm{P}^{\infty}) \cong Z[y], \quad CH^*(BZ/p) \cong H^{2*,*}(BZ/p) \cong Z[y]/(py)$$

with $deg(y) = (2,1)$. For product of these spaces

$$(1.2)\quad CH^*(\mathrm{P}^{\infty} \times ... \times \mathrm{P}^{\infty}) \cong Z[y_1, ..., y_n]$$

$$(1.3)\quad CH^*(BZ/p \times ... \times BZ/p) \cong Z[y_1, ..., y_n]/(py_1, ...py_n).$$

Here note that $CH^*(X) \not\cong H^{even}(X(\mathrm{C}))$ for the last case. Even if $H^*(X(\mathrm{C}))$ is generated by even dimensional elemets, there are cases that $CH^*(X) \not\cong H^*(X(\mathrm{C}))$, e.g., the K3-surfaces have the cohomology $H^2(X(\mathrm{C})) \cong Z^{22}$ but there is a K3-surface such that $CH^1(X) \cong Z^i$ for each $1 \le i \le 20$.

The Milnor K-theory is the graded ring $\oplus_n K_n^M(k)$ defined by $K_n^M(k) = (k^*)^{\otimes n}/J$ where the ideal J is generated by elements $a \otimes (1-a)$ for $a \in k^*$. Hence $K_0^M(k) = Z$ and by definition$K_1^M(k)$ is just the multiplicative group k^* but written additively in the ring $K_*^M(k)$. Hilbert's theorem 90, which is essentialy said that the Galois cohomology $H^1(G(k_s/k); k_s^*) = 0$, implies the isomorphism $K_1^M(k)/p \cong k^*/(k^*)^p \cong H^1(G(k_s/k); Z/p)$ for $1/p \in k$. Similarly we can define a map (the norm residue map) for any extension F of k of finite type

$$(BK)\qquad K_n^M(F)/p \to H^n(G(F_s/F); \mu_p^{\otimes n})$$

where $\mu_p^{\otimes n}$ is the discrete $G(F_s/F)$-module of n-th tensor power of the group of p-roots of 1.

The Bloch-Kato conjecture is that this map is an isomorphism for all field k and the Milnor conjecture is its $p = 2$ case. This conjecture is solved when $n = 2$ by Merkurjev-Susulin[Me-Su], and for $p = 2$ by Voevodsky [Vo1] by usig the motivic cohomology.

Notice that $H^n(G(k_s/k); \mu_p^{\otimes n}) \cong H_{et}^n(Spec(k), \mu_p^{\otimes n})$ the étale cohomology of the point. The étale cohomology $H_{et}^*(X; Z/p)$ has the properties ;

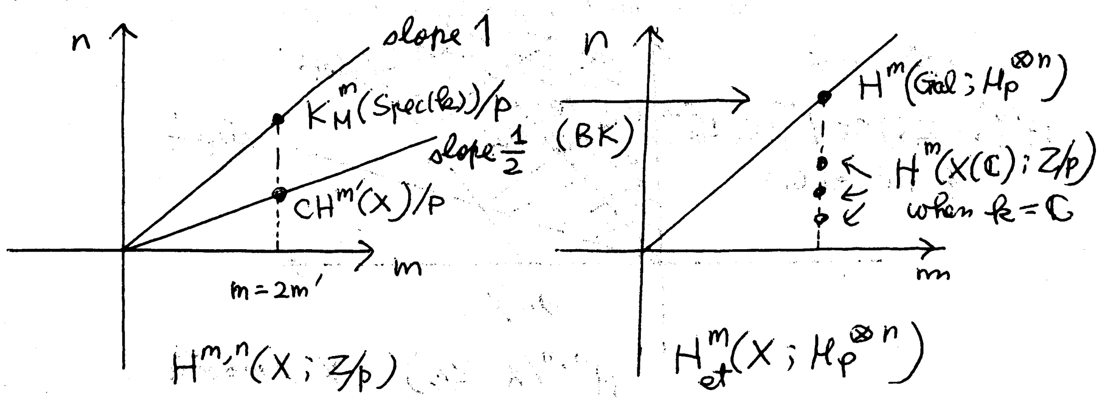
(E.1) If k contains a primitive p-th root of 1, then there is the additive isomorphism

$$H_{et}^m(X, \mu_p^{\otimes n}) \cong H_{et}^m(X; Z/p).$$

(E.2) For smooth X over $k = \mathrm{C}$,

$$H_{et}^m(X; Z/p^N) \cong H^m(X(\mathrm{C}); Z/p^N) \quad \textit{for all } N \ge 1.$$

The last cohomology is the usual mod p ordinary cohomology of C-rational point of X. Of course $H_{et}^*(Spec(\mathrm{C}); Z/p) \cong Z/p$. It is known that $K_*^M(\mathrm{R})/2 \cong H_{et}^*(Spec(\mathrm{R}); Z/2) \cong Z/2[\rho]$ with $deg(\rho) = 1$ for the real number field. Here $\rho = \{-1\} \in K_1^M(\mathrm{R}) = \mathrm{R}^*/\mathrm{R}^2$. Let F_v be a local field with residue field k_v of $ch(k_v) \ne 2$. Then $K_*^M(F_v)/2 \cong H_{et}^*(Spec(F_v); Z/2) \cong \Lambda(\alpha, \beta)$ with $deg(\alpha) = deg(\beta) = 1$. Thus we know $\oplus_m H^{m,m}(pt; Z/2)$ for these cases.



2. THE REALIZATION MAP

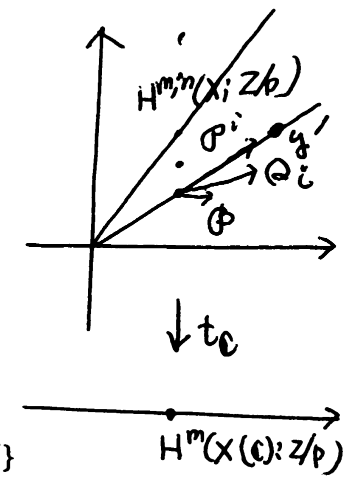
In this section we consider the relation to the usual ordinary cohomology. Let R be $\mathbb{Z}$ or $\mathbb{Z}/p$. The motivic cohomology has the following properties [Vo2].

(C1) $H^{*,*}(X;R)$ is a bigraded ring natural in X.

(C2) There are maps (realization maps)

$$t_{\mathbb{C}}^{m,n} : H^{m,n}(X;R) \to H^m(X(\mathbb{C});R)$$

which sum up $t_{\mathbb{C}}^{*,*} = \oplus_{m,n} t_{\mathbb{C}}^{m,n}$ the natural ring homomorphism.



(C3) There are (the Bockstein, the reduced powers) operations

$$\beta : H^{*,*}(X;\mathbb{Z}/p) \to H^{*+1,*}(X;\mathbb{Z}/p)$$

$$P^i : H^{*,*}(X;\mathbb{Z}/p) \to H^{*+2(p-1)i,*+(p-1)i}(X;\mathbb{Z}/p)$$

which commutes with the realization map $t_{\mathbb{C}}$.

(C4) For the projective space $\mathbb{P}^n$, there is an isomorphism

$$H^{*,*}(X \times \mathbb{P}^n/\mathbb{P}^{n-1};R) \cong H^{*,*}(X;R)\{1,y'\}$$

with $deg(y') = (2n,n)$ and $t_{\mathbb{C}}(y') \neq 0$.

Here we consider some examples. Recall $H^*(\mathbb{C}P^\infty = \mathbb{P}^\infty(\mathbb{C});\mathbb{Z}/p) \cong \mathbb{Z}/p[y]$, $deg(y) = 2$ and $H^*(B\mathbb{Z}/p(\mathbb{C}) = B\mathbb{Z}/p;\mathbb{Z}/p) \cong \mathbb{Z}/p[y] \otimes \Lambda(x)$ with $\beta x = y$ (when $p = 2$, $y = x^2$). From the above properties (C1), (C2), we easily see that $t_{\mathbb{C}}$ is epic for $X = \mathbb{P}^\infty$. Moreover there is $x' \in H^{1,1}(B\mathbb{Z}/p;\mathbb{Z}/p)$ such that $t_{\mathbb{C}}(x') = x$ and from (C2), we also see $t_{\mathbb{C}}$ is epic for $X = B\mathbb{Z}/p$.

To see these facts hold for other spaces, we recall the Lichtenbaum motivic cohomology [Vo2]. Lichtenbaum defined the similar cohomology $H_L^{*,*}(X;R)$ by using the étale topology, while $H^{*,*}(X;R)$ is defined by using Nisnevich topology. Since Nisnevich covers are some restricted étale covers, there is the natural map $H^{*,*}(X;R) \to H_L^{*,*}(X;R)$. We say that the condition $B(n,p)$ holds if

$$B(n,p) \quad : \quad H^{m,n}(X;Z_{(p)}) \cong H_L^{m,n}(X;Z_{(p)}) \quad \textit{for all } m \leq n+1$$

and all smooth X. The Beilinson-Lichtenbaum conjecture is that $B(n,p)$ holds for all n, p. It is proved that the $B(n,p)$ condition is equivalent the Bloch-Kato conjeture (BK) for degree n and prime p. Hence $B(n,p)$ holds for $n \leq 2$ or $p = 2$. Moreover Suslin-Voevodsky proves

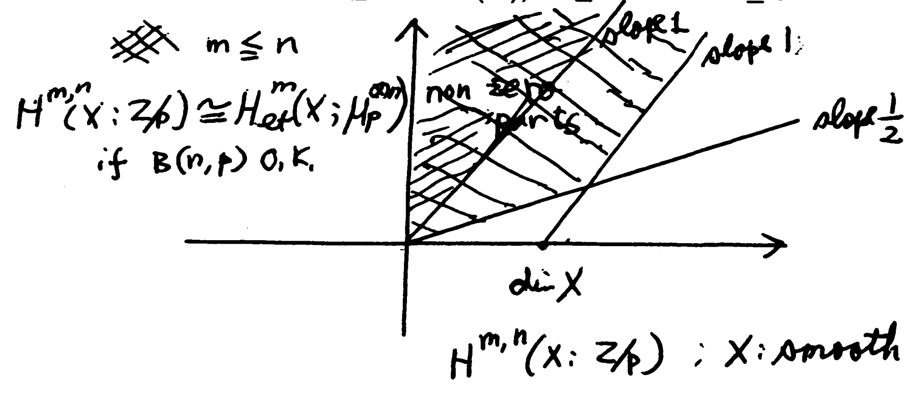
(L-E) If $1/p \in k$, then for all X,

$$H_L^{m,n}(X;\mathbb{Z}/p) \cong H_{et}^m(X;\mu_p^{\otimes n}).$$

Now we compute $H^{*,*}(pt = Spec(k);\mathbb{Z}/p)$. For a smooth X, it is known the following dimensional conditions.

(C5) For a smooth X, if $H^{m,n}(X;R) \not\cong 0$, then

$$m \leq n + dim(X),\ m \leq 2n \ \textit{and} \ m \geq 0.$$



Hereafter this paper,we assume that k contains a primitive p-th root of 1 and $B(n,p)$ holds for all n but $X = Spec(k)$. Then

$$H^{m,n}(pt;\mathbb{Z}/p) \cong H^m_{et}(pt;\mu_p^{\otimes n}) \cong H^m_{et}(pt;\mathbb{Z}/p) \qquad if\ m \le n$$

and $H^{m,n}(pt;\mathbb{Z}/p) \cong 0$ otherwise. Let $\tau \in H^{0,1}(pt;\mathbb{Z}/p)$ be the element corresponding a generator of $H^0_{et}(Spec(k);\mu_p) \cong H^0_{et}(Spec(k);\mathbb{Z}/p)$. Then we get the isomorphism

$$H^{*,*}(Spec(k);\mathbb{Z}/p) \cong H^*_{et}(Spec(k);\mathbb{Z}/p) \otimes \mathbb{Z}/p[\tau]$$

since $\tau : H^m_{et}(pt;\mu_p^{\otimes n}) \cong H^m_{et}(pt;\mu_p^{\otimes(n+1)})$. Inparticular, for the real number field $\mathbb{R}$ and a local field F_v with the residue field k_v of $ch(k_v) \neq 2$

$$(2.1) \qquad H^{*,*}(Spec(\mathbb{R});\mathbb{Z}/2) \cong \mathbb{Z}/2[\rho,\tau] \quad with\ deg(\rho) = (1,1)$$

$$(2.2) \quad H^{*,*}(Spec(F_v);\mathbb{Z}/2) \cong \mathbb{Z}/2[\tau] \otimes \Lambda(\alpha,\beta) \quad with\ deg(\alpha) = deg(\beta) = (1,1).$$

For $k = \mathbb{C}$, $B(n,p)$ condition holds for $X = Spec(\mathbb{C})$, indeed $K^M_n(\mathbb{C}) \cong 0$ for $n > 0$. Therefore

$$(2.3) \quad H^{*,*}(Spec(C);\mathbb{Z}/p) \cong \mathbb{Z}/p[\tau] \quad with\ deg(\tau) = (0,1).$$

When $k = \mathbb{C}$, if $B(n,p)$ condition holds for X, then it is immediate that

$$(2.4) \qquad [\tau^{-1}]H^{*,*}(X;\mathbb{Z}/p) \cong H^*(X(\mathbb{C});\mathbb{Z}/p) \otimes \mathbb{Z}/p[\tau,\tau^{-1}]$$

where the degree is defied by $deg(x) = (m,m)$ if $x \in H^m(X(\mathbb{C});\mathbb{Z}/p)$.

Next we compute cohomology of $\mathbb{P}^\infty$ and $B\mathbb{Z}/p$. For any (algebraic) map $f : X \to Y$ in the category Spc, we can construct the cofiber sequence

$$X \to Y \to cone(f) = Y/X$$

which induces the long exact sequence (Voevodsky [V2])

$$(2.5) \quad H^{*,*}(X;R) \leftarrow H^{*,*}(Y;R) \leftarrow H^{*,*}(Y/X : R) \leftarrow H^{*-1,*}(X;R).$$

In particular, we get the Mayer-Vietoris, Gysin and blow up long exact sequences.

By the cofiber sequence $\mathbb{P}^{n-1} \to \mathbb{P}^n \to \mathbb{P}^n/\mathbb{P}^{n-1}$ and (C4), we can inductively see that

$$(2.6) \quad H^{*,*}(\mathbb{P}^n;\mathbb{Z}/p) \cong H^{*,*}(pt;\mathbb{Z}/p) \otimes \mathbb{Z}/p[y]/(y^{n+1}) \quad with\ \ deg(y) = (2,1)$$

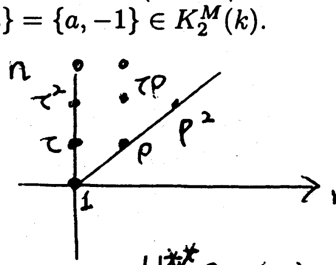
Since $B(1,p)$ is always holds, $H^{1,1}(L^n_p;\mathbb{Z}/p) \cong H^1(L^n_p;\mathbb{Z}/p)$. Hence there is the element $x' \in H^{1,1}(L^n_p;\mathbb{Z}/p)$ with $t_{\mathbb{C}}(x') = x \in H^1(L^n_p;\mathbb{Z}/p)$. The Lens space is identified with the sphere bundle associated with the line bundle

$$(\mathbb{A}^n - \{0\}) \times_{(\mathbb{A}-\{0\})} \mathbb{A} \to (\mathbb{A}^n - \{0\})/(\mathbb{A} - \{0\}) = \mathbb{P}^n.$$

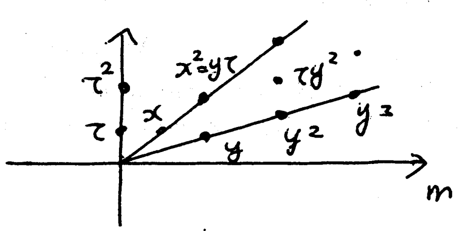
Where $(\mathbb{A}^n - \{0\}) \times_{(\mathbb{A}-\{0\})} \mathbb{A}$ is the identification such that $(z_i,z) \sim (a^{-1}z_i, a^pz) \in (\mathbb{A}^n - \{0\}) \times \mathbb{A}$. Hence we get the cofibering $L^n_p \to \mathbb{P}^n \xrightarrow{\times p} \mathbb{P}^n$. Thus we get the additive isomorphism $H^{*,*}(L^n_p;\mathbb{Z}/p) \cong H^{*,*}(\mathbb{P}^n;\mathbb{Z}/p)\{1,x\}$. This induces the ring isomorphism for $p = odd$

$$(2.7) \quad H^{*,*}(L^n_p;\mathbb{Z}/p) \cong \mathbb{Z}/p[y]/(y^{n+1}) \otimes \Lambda(x) \otimes H^{*,*}(pt;\mathbb{Z}/p) \quad with\ deg(x) = (1,1).$$

However note that when $p = 2$, we see $x^2 = y\tau + x\rho$ [Vo3] where $\rho \in H^{1,1}(pt;\mathbb{Z}/p) \cong k^*/k^{2*}$ represents -1. (hence $\rho = 0$ when $\sqrt{-1} \in k^*$.) This is proved by the wellknown facts $\{a,a\} = \{a,-1\} \in K^M_2(k)$.



$H^{*,*}(Spec(\mathbb{R}); \mathbb{Z}/2)$



$H^{*,*}(B\mathbb{Z}/2; \mathbb{Z}/2)$

$k = \mathbb{C}$

Let us say that a space X satisfies the Kunneth formula for a space Y if $H^{*,*}(X \times Y;\mathbb{Z}/p) \cong H^{*,*}(X;\mathbb{Z}/p) \otimes_{H^{*,*}(pt;\mathbb{Z}/p)} H^{*,*}(Y;\mathbb{Z}/p)$.
By the above cofiber sequences, we can easily see that P^{∞} and $B\mathbb{Z}/p$ satify the Kunneth formula for all spaces. In particular, we have the ring isomorhisms

$$(2.8) \quad H^{*,*}(\mathrm{P}^{\infty} \times ... \times \mathrm{P}^{\infty};\mathbb{Z}/p) \cong \mathbb{Z}/p[y_1,...,y_n] \otimes H^{*,*}(pt;\mathbb{Z}/p)$$

$$(2.9) \quad H^{*,*}(B\mathbb{Z}/p \times ... \times B\mathbb{Z}/p;\mathbb{Z}/p) \cong \mathbb{Z}/p[y_1,...,y_n] \otimes \Lambda(x_1,...,x_n) \otimes H^{*,*}(pt;\mathbb{Z}/p)$$

(when $p=2$, $x_i^2 = y_i\tau + x_i\rho$).

This fact is used to defined the reduced power operation P^i in (C3). Since the Sylow p subgroup of the symmetric group S_p of p-letters, is isomorphic to $\mathbb{Z}/p$, we know the isomorphism

$$H^*(BS_p;\mathbb{Z}/p) \cong H^*(B\mathbb{Z}/p;\mathbb{Z}/p)^{F_p^*} \cong \mathbb{Z}/p[Y] \otimes \Lambda(X)$$

with identifying $Y = y^{p-1}$ and $X = xy^{p-2}$. If X is smooth (and suppose p is odd for easy of arguments), we can define the reduced powers (of Chow rings) as follows. Consider maps

$$H^{2*,*}(X;\mathbb{Z}/p) \xrightarrow{i_!} H^{2p*,p*}(X^p \times_{S_p} ES_p) \xrightarrow{\Delta^*} H^*(X;\mathbb{Z}/p) \otimes_{H^{*,*}} H^{*,*}(BS_p;\mathbb{Z}/p)$$

where $i_!$ is the Gysin map for p-th external power, and Δ is the diagonal map. For $deg(x) = (2n,n)$, the reduced powers are defined as

$$(2.10) \quad \Delta^* i_!(x) = \sum P^i(x) \otimes Y^{n-i} + \beta P^i(x) \otimes XY^{n-i-1}.$$

Hence note $deg(P^i) = deg(Y^i) = deg(y^{i(p-1)}) = (2i(p-1), i(p-1))$.

Voevodsky defined $i_!$ for non smooth X also and by using suspensions maps, he defined reduced poweres for all degree elements in $H^{*,*}(X;\mathbb{Z}/p)$ for all X [Vo 3].

Moreover we can see (Ho-Kriz [H-K])

$$(2.11) \quad H^{*,*}(BGL_n;\mathbb{Z}/p) \cong \mathbb{Z}/p[c_1,...,c_n] \otimes H^{*,*}(pt;\mathbb{Z}/p)$$

where the Chern class c_i with $deg(c_i) = (2i,i)$ are identified with the elementary symmetric polynomial in $H^{*,*}(\mathrm{P}^{\infty} \times ... \times \mathrm{P}^{\infty};\mathbb{Z}/p)$. So we can define the Chern class $\rho^*(c_i) \in H^{2*,*}(BG;\mathbb{Z}/p)$ for each algebraic group G and for each representation $\rho: G \to GL_n$.

3. $H^{*,*}(X;\mathbb{Z}/p)/Ker(t_{\mathbb{C}})$ AND OPERATION Q_i

In this section we always assume that X is smooth and $k = \mathbb{C}$. Define a bidegree algebra by

$$(3.1) \qquad h^{*,*}(X;\mathbb{Z}/p) \;=\; \oplus_{m,n} H^{m,n}(X;\mathbb{Z}/p)/Ker(t_{\mathbb{C}}^{m,n}).$$

Suppose that $B(n,p)$ condition holds. By isomorphisms (B,p),(L-E),(E1) and (E2), we have

$$H^{n,n}(X;\mathbb{Z}/p) \cong H_L^{n,n}(X;\mathbb{Z}/p) \cong H_{et}^n(X;\mu_p^{\otimes n}) \cong H_{et}^n(X;\mathbb{Z}/p) \cong H^n(X(\mathbb{C});\mathbb{Z}/p).$$

The realization map $t_{\mathbb{C}}^{n,n}$ induces this isomorphism. Let $F_i = Im(t_{\mathbb{C}}^{*,i})$. Then $\cup_i F_i = H^*(X(\mathbb{C});\mathbb{Z}/p)$ and define the graded algebra $grH^*(X(\mathbb{C});\mathbb{Z}/p) = \oplus F_{i+1}/F_i$. Thus we get the additive isomorphism

$$h^{*,*}(X;\mathbb{Z}/p) \cong grH^*(X(\mathbb{C});\mathbb{Z}/p) \otimes \mathbb{Z}/p[\tau]$$

of bigraded rings. However the ring structures of both rings are different, in general. The cohomology $h^{*,*}(X;\mathbb{Z}/p)$ is isomorphic to a $\mathbb{Z}[\tau]$-subalgebra B of $H^*(X(\mathbb{C});\mathbb{Z}/p) \otimes \mathbb{Z}/p[\tau,\tau^{-1}]$

with $deg(x) = (|x|, |x|)$ such that $B[\tau^{-1}] \cong H^*(X(\mathbb{C}); \mathbb{Z}/p) \otimes \mathbb{Z}/p[\tau, \tau^{-1}]$. Namely there is a $\mathbb{Z}/p$-basis $\{a_I\}$ of $H^*(X(\mathbb{C}); \mathbb{Z}/p)$ such that $B = \mathbb{Z}/p\{\tau^{-t_I} a_I\} \otimes \mathbb{Z}/p[\tau]$ for some $t_I \geq 0$.

Here we recall the Milnor primitive operation $Q_i = [Q_{i-1}, P^{p^{i-1}}]$

$$Q_i : H^{*,*}(X; \mathbb{Z}/p) \to H^{*+2p^i-1, *+p^i-1}(X; \mathbb{Z}/p)$$

which is derivative, $Q_i(xy) = Q_i(x)y + xQ_i(y)$. Note also $Q_i(\tau) = 0$ by dimensional reason of $H^{*,*}(pt; \mathbb{Z}/p) \cong \mathbb{Z}/p[\tau]$.

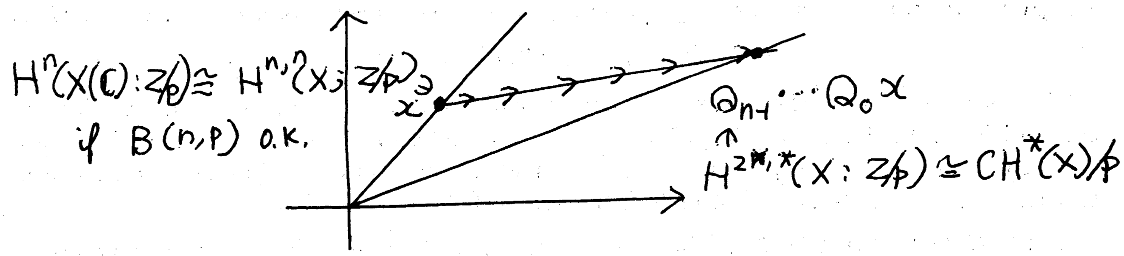
Lemma 3.1. *If $0 \neq Q_{i_1} ... Q_{i_s} x \in H^{2*,*}(X; \mathbb{Z}/p)$, then x is a $\mathbb{Z}/p[\tau]$-module generator.*

Proof. If $x = x'\tau$,then $\tau Q_{i_1} ... Q_{i_s}(x') \neq 0$. But $Q_{i_1} ... Q_{i_s}(x') = 0 \in H^{2*,*-1}(X; \mathbb{Z}/p)$ since $H^{m,n}(X; \mathbb{Z}/p) = 0$ for $m > 2n$. □

Define the weight by $w(x) = 2n - m$ for an element $x \in H^{m,n}(X; \mathbb{Z}/p)$ so that $w(x') = 0$ for $x' \in CH^*(X)/p$. Of course we get $w(xy) = w(x) + w(y)$, $w(P^i x) = w(x)$ and $w(Q_i(x)) = w(x) - 1$.

Corollary 3.2. *Suppose that $B(n,p)$ holds. If $x \in H^n(X(\mathbb{C}); \mathbb{Z}/p)$ and $Q_{i_1} ... Q_{i_n}(x) \neq 0$, then there is a $\mathbb{Z}/p[\tau]$-module generator $x' \in H^{n,n}(X; \mathbb{Z}/p)$ so that $t_{\mathbb{C}}(x') = x$ and for each $0 \leq k \leq n$, $Q_{i_1} ... Q_{i_k}(x')$ is also a $\mathbb{Z}/p[\tau]$- module generator of $H^{*,*}(X; \mathbb{Z}/p)$.*

Proof. By $B(n,p)$ condition, $t_{\mathbb{C}}^{n,n} : H^{n,n}(X; \mathbb{Z}/p) \cong H^n(X(\mathbb{C}); \mathbb{Z}/p)$. Hence there is an element $x' \in H^{n,n}(X; \mathbb{Z}/p)$ with $t_{\mathbb{C}}(x') = x$. This means $w(x') = n$ and $w(Q_{i_1} ... Q_{i_n}(x)) = 0$. From the above lemma, we get the corollary. □



Now we consider the examples. The mod 2 cohomology of $BO(n)$ is $H^*(BO(n); \mathbb{Z}/2) \cong \mathbb{Z}/2[w_1, ..., w_n]$ where the Stiefel-Whiteney class w_i restricts the elementary symmetric polynomial in $H^*(B(\mathbb{Z}/2)^n; \mathbb{Z}/2) \cong \mathbb{Z}/2[x_1, ..., x_n]$. Each element w_i^2 is represented by Chern class c_i of the induced representation $O(n) \subset U(n)$. Hence $c_i \in CH^*(BS(n); \mathbb{Z}/2) = H^{2*,*}(BO(n); \mathbb{Z}/2)$.

Proposition 3.3. *$h^{*,*}(BO(n); \mathbb{Z}/p) \supset \mathbb{Z}/2[c_1, ..., c_n] \otimes \Delta(w_1, ..., w_n) \otimes \mathbb{Z}/2[\tau]$ where $deg(c_i) = (2i, i)$, $deg(w_i) = (i, i)$ and $w_i^2 = \tau^i c_i$.*

Since $Q_{i-1} ... Q_0(w_i) \neq 0$, each w_i is a $\mathbb{Z}/2[\tau]$-module generator. However even $h^{*,*}(BO(n); \mathbb{Z}/2)$ seems very complicated. Consider the case $X = BO(3)$. The cohomology operations act by

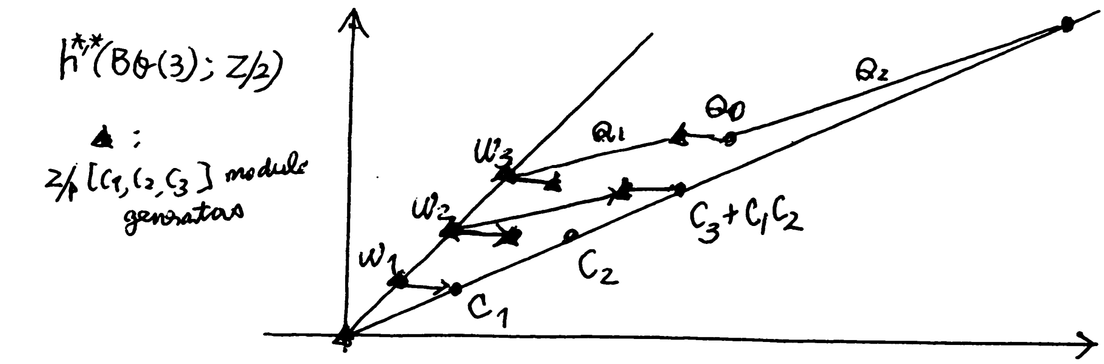
$$w_2 \xrightarrow{Sq^1} w_1 w_2 + w_3 \xrightarrow{Sq^2} w_2 w_1^3 + w_1^2 w_3 + w_1 w_2^2 + w_2 w_3 \xrightarrow{Sq^1} w_1^2 w_2^2 + w_3^2$$

$$w_3 \xrightarrow{Sq^1} w_3 w_1 \xrightarrow{Sq^2} w_1 w_2 w_3$$

Theorem 3.4. *There is the isomorphism*

$$h^{*,*}(BO(3); \mathbb{Z}/2) \cong \mathbb{Z}/2[c_1, c_2, c_3]\{1, w_1, w_2, Q_0 w_2, Q_1 w_2, w_3, Q_0 w_3, Q_1 w_3\} \otimes \mathbb{Z}/2[\tau].$$

where $Q_0 w_2 = \tau^{-1}(w_1 w_2 + w_3)$, ...



W.S.Wilson ([W],[K-Y]) found a good $Q(i) = \Lambda(Q_0, ..., Q_i)$-module decomposition for $X = BO(n)$, namely,

$$(3.2) \qquad H^*(X; Z/2) = \oplus_{-1} Q(i)G_i \quad with \quad Q_0...Q_iG_i \in t_C(CH^*(X)).$$

Here G_{k-1} is quite complicated , namely, it is generated by symmetric functions

$$\Sigma x_1^{2i_1+1}...x_k^{2i_k+1}x_{k+1}^{2j_1}...x_{k+q}^{2j_q}, \quad k+q \leq n,$$

with $0 \leq i_1 \leq ... \leq i_k$ and $0 \leq j_1 \leq ... \leq j_q$; and if the number of j equal to j_u is odd, then there is some $s \leq k$ such that $2i_s + 2^s < 2j_u < 2i_s + 2^{s+1}$.

Then $w(G_i) \geq i$ in $h^{*,*}(X; Z/p)$, that means

Proposition 3.5. *Givenig the weight by $w(G_i) = i+1$, we have the incusion for $X = BO(n)$*

$$h^{*,*}(X; Z/2) \subset (\oplus_i Q(i)G_i) \otimes Z/2[\tau].$$

One problem is that the above inclusion is really isomorphism or not. The similar decomposition holds for $X = (BZ/p)^n$ and the above inclusion is an isomorphism. The case $X = BO(3)$ is also isomorphism. Since the direct decomposition of $BO(3)$ is complicated to write, we only write here that of $SO(3)$ since $O(3) \cong SO(3) \times Z/2$.

$$(3.3) \quad H^*(BSO(3); Z/2) \cong Z/2[w_2, w_3] \cong Z/2[c_2, c_3]\{1, w_2, w_3 = Q_0w_2, w_2w_3 = Q_1w_2\}$$

$$\cong Z/2[c_2, c_3]\{w_2, Q_0w_2, Q_1w_2, c_3 = Q_0Q_1w_2\} \oplus Z/2[c_2]$$

$$\cong Z/2[c_2, c_3]Q(1)\{w_2\} \oplus Z/2[c_2].$$

Since there is the isomorphism $O(2n+1) \cong SO(2n+1) \times Z/2$, the cohomology of $BSO(2n+1)$ is reduced from that of $BO(2n+1)$. However note that the situation for $BO(2n)$ is different.

The extraspecial 2-group 2_+^{1+2n} is the n-th central product of the dihedral group D_8 of order 8. It has a central extension

$$(3.4) \quad 0 \to Z/2 \to G \to V = \oplus^{2n} Z/2 \to 0$$

Let $H^*(BV; Z/2) \cong Z/2[x_1, ..., x_{2n}]$. Then Quillen proved [Q2]

$$(3.5) \qquad H^*(BG; Z/2) \cong Z/2[x_1, ..., x_{2n}]/(f, Q_0f, ..., Q_{n-2}f) \otimes Z/2[w_{2^n}].$$

Here w_{2^n} is the Stiefel-Whiteney class of the real 2^n dimensional irreducible representation restricting non zero on the center and $f = \sum_i x_{2i-1}x_{2i} \in H^2(BV; Z/2)$ represents the central extension (3.4).

Leting $y_i = x_i^2$ in $H^*(BG; Z/2)$, we can write $\quad f^2 = \sum y_{2i-1}y_{2i}$,

$$(Q_{k-1}f)^2 = Q_0Q_kf = \sum y_{2i-1}^{2^k}y_{2i} - y_{2i-1}y_{2i}^{2^k}$$

$$Q_{k-1}f = \sum y_{2i-1}^{2^{k-1}}x_{2i} - x_{2i-1}y_{2i}^{2^{k-1}}.$$

Now we consider in the motivic cohomology $H^{*,*}(BG;\mathbb{Z}/2)$ and change $y_i = \tau^{-1}x_i^2$. Since $f = 0 \in H^{2,2}(BG;\mathbb{Z}/2)$, we can see that $Q_{k-1}f = 0$ and $Q_kQ_0(f) = 0$ also in $H^{*,*}(BG;\mathbb{Z}/2)$. However for general n, $\sum y_{2i}y_{2i-1} \neq 0$ in $H^{*,*}(BG;\mathbb{Z}/2)$. Let

$$(3.6) \qquad A = (\mathbb{Z}/2[y_1,...,y_{2n},c_{2^n}]/(Q_0Q_kf,...,Q_0Q_nf)$$
$$\otimes\Delta(x_1,...,x_2,w_{2^n})/(f,Q_0f,..,Q_{n-2}f))\otimes\mathbb{Z}/2[\tau].$$

Lemma 3.6. *For $G = 2_+^{1+2n}$, there is a map $A \to H^{*,*}(BG;\mathbb{Z}/2)$ which induces the injection $A/(f^2) \subset h^{*,*}(BG;\mathbb{Z}/2)$.*

When $m = 0,1,-1$ *mod* 8 and $m > 0$, we say that *Spin*(m) is *real type* [Q2]. When *Spin*(m) is real type, from Quillen, we know that $H^*(BSpin(m);\mathbb{Z}/2) \subset H^*(BG;\mathbb{Z}/2)$ where $G = 2_+^{2h+1}$, and h is the Hurwitz number (for details see [Q2]).

Corollary 3.7. *Let $G = Spin(m)$ be real type and the Hurwutz number h, and let*

$$A = (\mathbb{Z}/2[c_2,c_3,,...,c_m,c_{2^h}]/((Q_1Q_0w_2),...,(Q_hQ_0w_2)$$
$$\otimes\Delta(w_2,...,w_m,w_{2^h})/(w_2,Q_0w_2,...,Q_{h-2}w_2))\otimes\mathbb{Z}/2[\tau]$$

where $w_i, i \leq m$ (resp.w_{2^h}) is the Stiefel-Whitney class of the usual $SO(m)$ representation (resp. of the irreducible 2^h-dimensional spin representation). Then we have a map $A \to H^{*,*}(BG;\mathbb{Z}/2)$ which induces the injection $A/(c_2) \subset h^{*,*}(BG;\mathbb{Z}/2)$.

We study *Spin*(7) and the exceptional Lie group G_2. The cohomology of G_2 is given by $H^*(BG_2;\mathbb{Z}/2) \cong \mathbb{Z}/2[w_4,w_6,w_7]$ where w_i is the Siefel-Whitney class of the inclusion $G_2 \subset SO(7)$. The cohomology $H^*(BSpin(7);\mathbb{Z}/2) \cong H^*(BG_2;\mathbb{Z}/2)\otimes\mathbb{Z}/2[w_8]$.

Corollary 3.8. *Let $A = \mathbb{Z}/2[c_2,c_4,c_6,c_7]\otimes\Delta(w_4,w_6,w_7)\otimes\mathbb{Z}/2[\tau]$. Then there is the map $A \to H^{*,*}(BG_2;\mathbb{Z}/2)$ which induces the injection $A/(c_2) \subset h^{*,*}(BG_2;\mathbb{Z}/2)$. Similar facts hold for $BSpin(7)$ tensoring $\mathbb{Z}/2[c_8]$.*

The cohomology operations are given

$$w_4 \xrightarrow{Sq^2} w_6 \xrightarrow{Sq^1} w_7 \xrightarrow{Sq^4} w_4w_7 \xrightarrow{Sq^2} w_7w_6 \xrightarrow{Sq^1} w_7^2$$

$$Q_1Q_0(w_4w_6) = w_7^2, \quad Q_2Q_1Q_0(w_4w_6w_7) = w_7^4.$$

Proposition 3.9. *Let $w(w_4) = 2, w(w_{(4,6)}) = 2$ and $w(w_{(4,6,7)}) = 3$ with $t_{\mathbb{C}}(w_{(i_1,...,i_n)}) = w_{i_1}...w_{i_n}$. Then we have the injection*

$$h^{*,*}(BG_2;\mathbb{Z}/2) \subset \mathbb{Z}/2[c_4,c_6,c_7]$$
$$\otimes\mathbb{Z}/2\{1,w_4,Sq^2w_4,Q_1w_4,Q_2w_4,Sq^2Q_2w_4,w_{(4,6)},w_{(4,6,7)}\}\otimes\mathbb{Z}/p[\tau].$$

Remark. If $t_{\mathbb{C}}^{4,3}\otimes Q$ is epic, then we can take $w_4 \in h^{4,3}(BG;\mathbb{Z}/2)$, *i.e.*, $w(w_4) = 2$.
The kernel $Ker(t_{\mathbb{C}})^{2*,*}$ is not so big for $X = BG_2$. Indeed, it is known that

$$CH^*(BG_2) \cong Z_{(2)}[c_2,c_4,c_6,c_7]/(2^r(c_2^2-4c_4),2c_7,c_2c_7), \quad \textit{for some } r \geq 0.$$

The cohomology operations are given in $H^*(BSO(7);\mathbb{Z}/2)$

$$Q_1Q_0w_2 = w_3^2, \quad Q_2Q_0w_2 = w_5^2, \quad Q_3Q_0w_2 = w_7^2w_2^2 + w_6^2w_3^2 + w_5^2w_4^2.$$

Hence we have $c_3 = 0, \quad c_5 = 0 \quad c_2c_7 = 0$ in $CH^*(BG_2)$ but $c_2 \neq 0$.

From here we consider the case $p = odd$. One of the easist examples is the case $G = PGL_3$ and $p = 3$. The mod 3 cohomology is given by ([K-Y],[Vel])

$$(\mathbb{Z}/3[y_2]\{y^2\} \oplus \mathbb{Z}/3\{1, y_2, y_3, y_7\}[y_8]) \otimes \mathbb{Z}/3[y_{12}]$$

It is known that y_2^2, y_2^3, y_8^2 and y_{12} are represented by Chern classes. Moreover $Q_1Q_0(y_2) = y_8$. Hence these elements are in the Chow ring, namely,

$$h^{2*,*}(BPGL_3; \mathbb{Z}/3) \cong (\mathbb{Z}/3[y_2]\{y_2^2\} \oplus \mathbb{Z}/3[y_8]) \otimes \mathbb{Z}/3[y_{12}].$$

The cohomology operations are given

$$y_2 \xrightarrow{\beta} y_3 \xrightarrow{P^1} y_7 \xrightarrow{\beta} y_8$$

Thus we get $h^{*,*}(PGL_3; \mathbb{Z}/3)$ completely.

Theorem 3.10.

$$h^{*,*}(BPGL_3; \mathbb{Z}/3) \cong (\mathbb{Z}/3[y_2]\{y^2\} \oplus \mathbb{Z}/3\{1\} \oplus \mathbb{Z}/3[y_8] \otimes Q(1)\{y_2\}) \otimes \mathbb{Z}/3[y_{12}] \otimes \mathbb{Z}/3[\tau]$$

Next consider the extraspecial p-group $G = p_+^{1+2n}$. When $n > 2$, even the cohomology ring $H^*(BG(C); \mathbb{Z}/p)$ are unknown, while it contains the subring

$$B = \mathbb{Z}/p[y_1, ..., y_{2n}, c_{p^n}]/(Q_1Q_0f, ...Q_nQ_0f).$$

where $f = \sum^n x_{2i-1}x_{2i}$ for $\beta x_i = y_i$ and $Q_kQ_0f = \sum y_{2i-1}y_{2i}^{p^k} - y_{2i-1}^{p^k}y_{2i}$ Since $f = 0 \in H^{2,2}(BG; \mathbb{Z}/p)$, we have

Proposition 3.11. *Let $G = p_+^{1+2n}$ and $A = B \otimes \mathbb{Z}/p[\tau]$. Then there is an injection $A \subset H^{*,*}(BG; \mathbb{Z}/p)$*

We consider the case $n = 1$ here. Let us write $E = p_+^{1+2}$. The ordinary cohomology is known by Lewis [L], [Te-Y3], namely,

$$H^{even}(BE)/p \cong (\mathbb{Z}/p[y_1, y_2]/(y_1^py_2 - y_1y_2^p) \oplus \mathbb{Z}/p\{c_2, ..., c_{p-1}\}) \otimes \mathbb{Z}/p[c_p].$$

$$H^{odd}(BE) \cong \mathbb{Z}/p[y_1, y_2, c_p]\{a_1, a_2\}/(y_1a_2 - y_2a_1, y_1^pa_2 - y_2^pa_1) \quad |a_i| = 3.$$

Theorem 3.12.

$$h^{*,*}(BE; \mathbb{Z}/p) \cong (\{1, \partial^{-1}\}(H^*(BE)/p) - \{\partial^{-1}1\}) \otimes \mathbb{Z}/p[\tau]$$

where $w(H^{even}(BE)/p) = 0, w(H^{odd}(BE)) = 1$ and ∂_p^{-1} ascents the weight one.

Proof. Since all elements in $H^{even}(BE)$ are generated by Chern classes, we have the isomorphism $h^{2*,*}(BG; \mathbb{Z}/3) \cong H^{even}(BE)/p$. We know $H^{odd}(BE; \mathbb{Z}/p)$ is generated as a $H^{even}(BE)/p$-module by two elements a_1, a_2 such that $Q_1a_i = y_ic_p$ [Te-Y3].

The mod p-cohomology is written additively $H^*(BE; \mathbb{Z}/p) \cong \{1, \partial_p^{-1}\}H^*(BE)/p$. Here ∂_p is the (higher) Bockstein. All elements in $H^{odd}(BE)$ are just p-torsion and we can take $a_i' \in H^2(BE; \mathbb{Z}/p)$ such that $\beta(a_i') = a_i$. Thus we take $a_i' \in H^{2,2}(BE; \mathbb{Z}/p)$ so that $a_i \in H^{3,2}(BE; \mathbb{Z}/p)$.

Next consider elements $x = \partial_p^{-1}(y)$, $y \in H^{even}(BE)/p$. If $y \in (Ideal(y_1, y_2))$, then $\partial_p^{-1}(y) = \sum x_ib_i$ for $b_i \in H^{even}(BE)/p$, and hence we can take $w(\partial_p^{-1}(y)) = 1$. For other elements $y = c_ic$ with $c \in \mathbb{Z}/p[c_p]$, we can prove ([Ly]) that thse elements are represented by transfer from a subgroup isomorphic to $\mathbb{Z}/p \times \mathbb{Z}/p$. Therefore we can also prove that $w(\partial_p^{-1}(y)) = 1$. Thus we complete the proof. □

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